Chi-square distribution

Suppose $Z_1, \ldots, Z_n$ are independent random variables having the standard normal distribution. Then, $Y = \sum_{i=1}^{n} Z_i^2$ has the chi-square (χ^2) distribution with n degrees of freedom.

$$Y = Z_1^2 + \ldots + Z_n^2 \sim \chi_n^2 \Rightarrow \mu[Y] = E[Z_1^2] + \ldots + E[Z_n^2]$$
$$= 1 + \ldots + 1 = n$$

$$\mu_Y = E[Y] = n \qquad \int_{-\infty}^{\infty} x p(x)\, dx \to \mu$$

$$\int (x-\mu)^2 p(x)\, dx \to \sigma^2$$

and $\sigma_Y^2 = \mathrm{Var}[Y] = 2n$

Thm If $\bar{X}$ and S^2 are the mean and the variance of a random sample of size n from a normal population with mean μ and std deviation σ, then:

(1) $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

are independent.

(think of stock prices with daily mean 1, sampled hourly).

(2) The random variable

$\frac{(n-1)s^2}{\sigma^2}$ has the chi-square distribution with n-1 degrees of freedom

This follows from the identity:

$$\sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2 = \frac{(n-1)s^2}{\sigma^2} + \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$$\Rightarrow \quad \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

This has direct application to estimation of the population variance σ^2 from a normal distribution:

$$P\left[\underbrace{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}}_{\chi^2_R} < \sigma^2 < \underbrace{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}_{\chi^2_L} \right] = 1-\alpha$$

Ex] (from HW) In 16 test runs, the gas consumption of an engine has a std deviation of 2.2 gallons. Construct a 99% confidence interval for σ^2 (the true variability of gas consumption)

$n = 16, \quad s = 2.2, \quad \alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,\,n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\,n-1}}$$

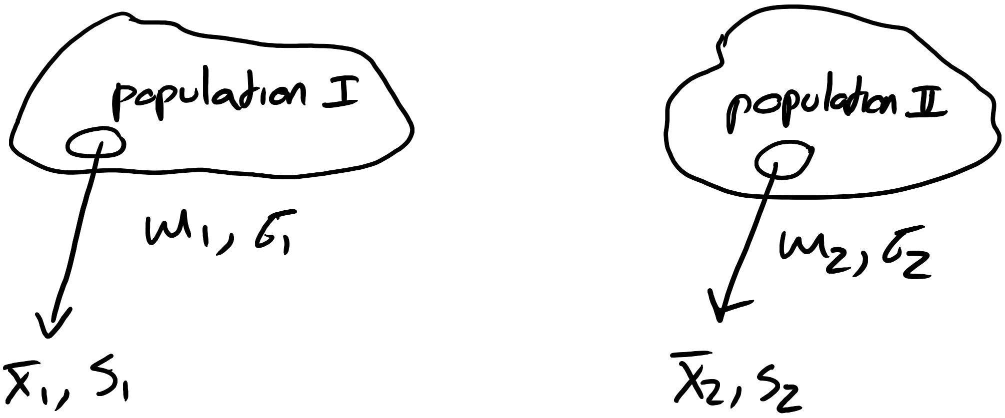
$$\chi^2_{0.005,\,15} = 32.801, \quad \chi^2_{0.995,\,15} = 4.601$$

$$\Rightarrow \quad \frac{15(2.2)^2}{32.801} < \sigma^2 < \frac{15(2.2)^2}{4.601}$$

$\Rightarrow 2.21 < \sigma^2 < 15.78$ is a 99% conf. interval for σ^2

$\Rightarrow 1.49 < \sigma < 3.97$

Estimation of differences between means



want to judge difference $\mu_1 - \mu_2$ between two population means

$\Rightarrow$ If $\bar{x}_1$ and $\bar{x}_2$ are the values of the means of independent random samples of size n_1 and n_2 from normal populations (or if, $n_1, n_2 \geq 30$ from any populations):

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1) \quad \text{(std normal distribution)}$$

$$\Rightarrow P(-z_{\alpha/2} < z < z_{\alpha/2}) = 1 - \alpha$$

$$(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < (\mu_1 - \mu_2)$$

$$< (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad \text{is } (1-\alpha)\cdot 100\% \text{ CI}$$

Ex) Construct a 94% confidence interval for the difference between the mean lifetimes of two kinds of light bulbs, given that a random sample of 40 bulbs of first kind lasted on avg 418 hours and a random sample of 50 bulbs of second kind lasted on avg 402 hrs of use. $\sigma_1 = 26$, $\sigma_2 = 22$.

$\Rightarrow \alpha = 0.06 \Rightarrow \frac{\alpha}{2} = 0.03 \Rightarrow z_{0.03} = 1.88$

(recall that $P(|Z| < z_{\alpha/2}) = 2P(0 < Z < z_{\alpha/2}) = 1-\alpha$)

$$\Rightarrow (418-402) - 1.88\sqrt{\frac{26^2}{40} + \frac{22^2}{50}} < (\mu_1 - \mu_2)$$

$$< (418-402) + 1.88\sqrt{\frac{26^2}{40} + \frac{22^2}{50}}$$

$\Rightarrow \quad 6.3 < \mu_1 - \mu_2 < 25.7$ with 94% confidence

difference in expected light bulb lifetimes

Hypothesis Testing (Chapter 8)

H_0 (null hypothesis): A theory about the values of one or more population parameters. The theory represents the status quo, which we accept until proven false.

H_1 (or H_a, alternative research hypothesis): A theory that contradicts the null hypothesis.

H_1 will be accepted only when sufficient evidence exists to establish the truth.

test statistic : A sample statistic used to decide whether to reject the null hypothesis.

rejection region : the numerical values of the test statistic for which the null hypothesis will be rejected.

Ex/ Assume 100 babies are born to 100 couples using the XSORT method of gender selection that is claimed to make girls more likely. Suppose we observe 58 girls in 100 babies. Write the hypothesis to test the claim "with XSORT treatment, the proportion of girls is greater than the 50% which occurs without treatment".

$H_0 : p = 0.5$ (null hypothesis)

$H_1 : p > 0.5$ (alternate hypothesis)

select significance level α based on a seriousness of a Type I error.

Type I error: Deciding that the null hypothesis is false when in fact it is true.

The risk of making a Type I error is denoted by α.

Ex] Suppose city wants to determine if a manufacturer's water pipe meets safety specifications.

requirement: strength of pipe be more than 2400 pounds per foot of pipe length.

H_0: $\mu \le 2400$ (product does not meet specifications)

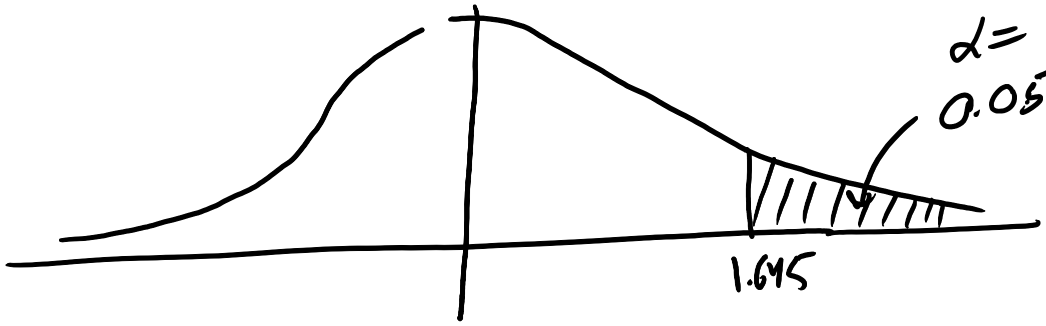
H_1: $\mu > 2400$ (product does meet specs)

Suppose sample of $n=50$ pipes is taken. We find that $\bar{x}=2460$, $s=200$

$$z = \frac{\bar{x}-\mu}{\sigma_{\bar{x}}} = \frac{\bar{x}-2400}{\sigma/\sqrt{n}} \approx \frac{\bar{x}-2400}{s/\sqrt{n}}$$

$$\Rightarrow \quad z \approx \frac{2460-2400}{200/\sqrt{50}} \approx \frac{60}{28.28} \approx 2.12$$

Thus, sample mean lies 2.12σ above hypothesized value of $(\mu = 2400)$.



$Z > 1.645$

Based on this α value we reject the null hypothesis.

Tests of Hypothesis

We utilize sample information to test whether a population parameter is less than, equal to, or greater than a specified value.

Just as with confidence interval estimation, it is important to state the measurement of the reliability of the inference. An inference without a measure of reliability is little more than a guess.

Ex) Measuring water pipe safety (from last time)

μ: mean strength of pipe (lb/ft) Safety requirement $\mu > 2400$

Null Hypothesis H_0: $\mu \le 2400$

Alternate Research Hypothesis H_1: $\mu > 2400$

"Convincing" evidence in favor of the alternative hypothesis will exist, when the value of $\bar{x}$ exceeds 2400 by an amount that cannot be readily attributed to sampling variability.

Test statistic: $n = 50 > 30$ by CLT

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \frac{\bar{x} - 2400}{\sigma/\sqrt{n}} \sim N(0,1)$$

Suppose the city tests 50 sewer pipes. They find:

$\bar{x} = 2460$

Suppose σ is known to be 200 (from the manufacturer).

$$\Rightarrow z = \frac{\bar{x} - \mu_{H_0}}{\sigma/\sqrt{n}} = \frac{\bar{x} - 2400}{\sigma/\sqrt{n}} = \frac{2460 - 2400}{200/\sqrt{50}}$$

$$= \frac{60}{28.28} \approx \underline{\underline{2.12}}$$

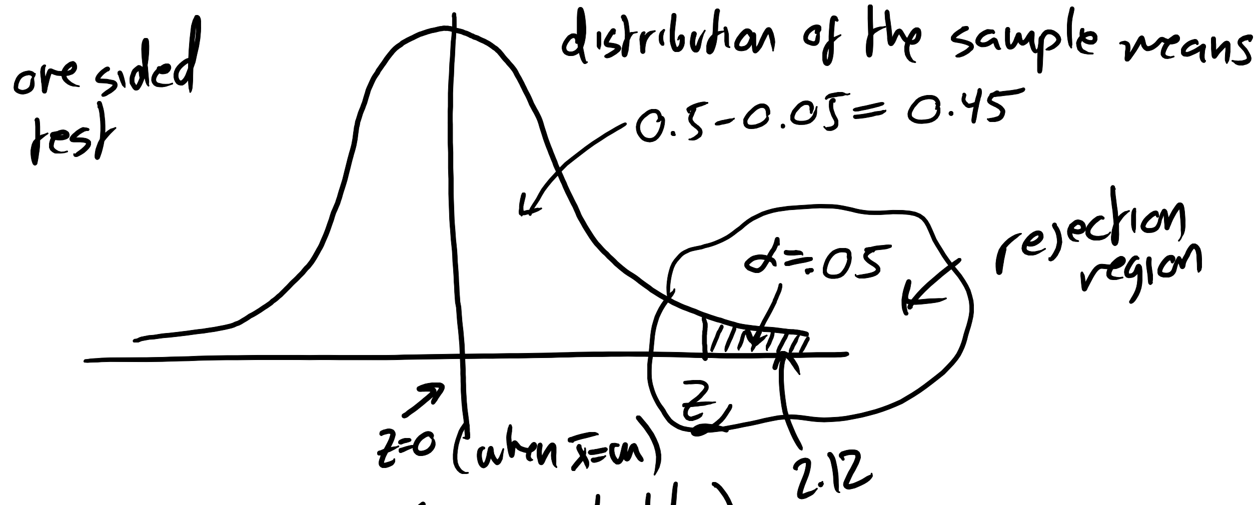
(2.12 × std dev of all possible sample means)

Thus, the sample mean lies $2.12\,\sigma_{\bar{x}}$ above the hypothesized mean value μ of 2400.

<u>Pick rejection region</u> The values of the test statistic for which we will reject the null hypothesis.

Pick $\alpha = .05$

Type I error: deciding in favor of H_1 when H_0 is true.



$z_\alpha = 1.645$ (from table)

$\Rightarrow \alpha = P(z > 1.645$ when in fact $\mu = 2400) = .05$

That is, $\alpha = P($Type I error$)$
$= P($rejecting H_0 when in fact its true$)$.

Summary

Test statistic: $$z = \frac{\bar{x} - 2400}{\sigma_{\bar{x}}}$$

Rejection region: $z > 1.645$, which corresponds to $\alpha = 0.05$.

For our sample, $z \approx 2.12 > 1.645$
Thus, the test statistic value falls in the rejection region. Thus, we reject H_0.

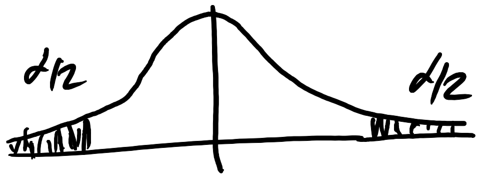
What is the probability that our statistical test could lead us to reject the null hypothesis when in fact it is true?

The answer is $\alpha = 0.05$.

We selected the level of risk, α, of making a Type I error when we constructed the test.

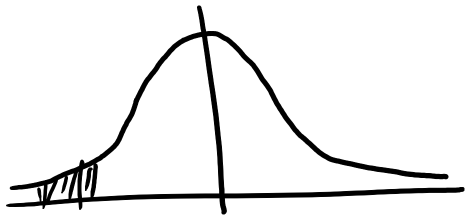
<u>Types of Hypothesis Tests</u> ((differ by rejection (critical) region)

(Two tailed, left tailed, right tailed).



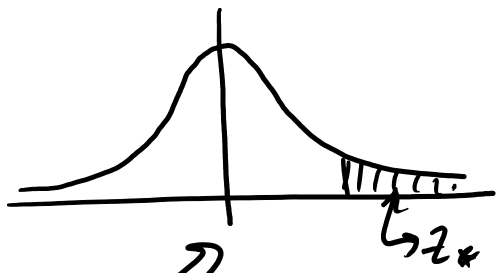
two tailed tests

Ex) $H_a: \mu \neq 2400$



left tailed test

Ex) $H_a: \mu < 2400$



right tailed test

Ex) $H_a: \mu > 2400$

$z=0$ corresponds to $\bar{x} = \mu$

$z = z_*$ corresponds to $\bar{x} \gg \mu$ $(\bar{x} > \mu + z_* \sigma_{\bar{x}})$

Ex) Neurologist testing the effects of the drug on response time by injecting 100 rats with a unit dose of the drug.

mean response time for rats not injected with the drug is $\mu = 1.2$ seconds.

wish to test whether the mean response time for drug induced rats differs from 1.2 seconds.

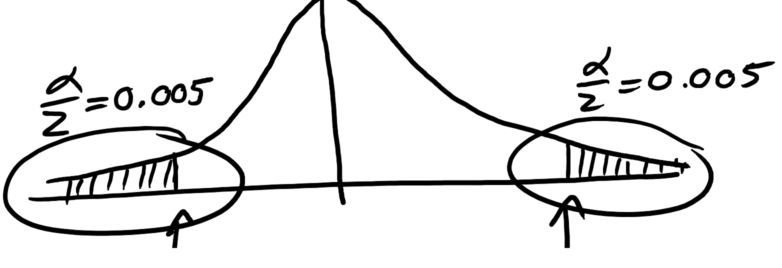
$\Rightarrow \left\{ \begin{array}{ll} H_0: \mu = 1.2 & \text{(drug induced rats have same reaction time as non-drug induced rats)} \\ H_1: \mu \neq 1.2 & \text{(different reaction times)} \end{array} \right.$

We conduct a <u>two-tailed</u> statistical test.

Set up test of hypothesis using $\alpha = 0.01$.

Test statistic $z = \dfrac{\bar{x} - 1.2}{\sigma_{\bar{x}}} = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$\alpha = 0.01 \Rightarrow \alpha/2 = 0.005$ is placed in each tail



$\bar{x} = 10$

$\sigma = 2 \quad n = 100$

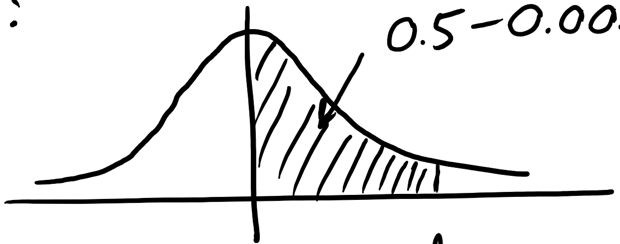
$z = \dfrac{10 - 1.2}{2/10} \cdots$

$z' = -2.575$ $z' = 2.575$

$$\boxed{z = \frac{8.8}{0.2} > 2.575}$$

How?

$0.5 - 0.005 = 0.495$



Find z from table corresponding to 0.495 area.

Rejection region: reject null hypothesis if your sample has a mean $\bar{x}$ that is $> 2.575\sigma_{\bar{x}}$ or $< 2.575\sigma_{\bar{x}}$ from the expected value.

$z < -2.575$ or $z > 2.575$

Exam #

$$P(x > 3) = 1 - P(x \leq 3)$$

$$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3)]$$

$$P(x_{poisson} = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$P(x_{binom} = k) \approx P(x_{poisson} = k) \text{ with } \lambda = np$$

$= 2000(0.001)$
$= 2$

$$P(x_{binom} > 3) \approx 1 - \sum_{k=0}^{3} \frac{\lambda^k e^{-\lambda}}{k!}$$

③ (b) $P(|\bar{x}-\mu| < 7) = ?$

probability that the error in estimation of μ by $\bar{x}$ is less than 7.

$$Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \qquad P(|Z| < Z_{\alpha/2}) = 1-\alpha$$

$$P\left(\frac{|\bar{x}-\mu|}{\sigma/\sqrt{n}} < Z_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\Big(|\bar{x}-\mu| < \underbrace{Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{\text{error in estimation}}\Big) = 1-\alpha$$

$$\Rightarrow Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 7 \Rightarrow Z_{\alpha/2}\frac{20}{\sqrt{100}} = 7$$

$$\Rightarrow Z_{\alpha/2} = \frac{7}{2} = 3.5 \Rightarrow \text{what is } \alpha?$$

$$P(|Z| < Z_{\alpha/2}) = 1-\alpha = 2P(0 < Z < Z_{\alpha/2})$$
$$= 2P(0 < Z < 3.5)$$
$$= 2(.4998) = 0.9996$$

Hypothesis Testing (continued)

Ex) In flipping an ordinary coin, the probabilities of heads and tails are both $\frac{1}{2}$.

Suppose a coin collector has a rare dime whose prob. of heads and tails are $\frac{3}{4}$, $\frac{1}{4}$.

Imagine the collector misplaces the dime. Then he finds another dime identical in appearance and wonders if it is his rare dime.

Let $\theta_0, \theta_1, \theta$ be the probabilities of <u>heads</u> using: rare dime, an ordinary dime, the dime collector has found.

$$H_0: \theta = \theta_0 = \frac{3}{4} \quad \text{(special coin)}$$

(null Hypothesis is that he has found the rare dime)

$$H_1: \theta = \theta_1 = \frac{1}{2} \quad \text{(regular coin)}$$

(alternate Hypothesis is that the dime he has found is an ordinary one).

<u>Type I error:</u> the error of rejecting the H_0 hypothesis when the null hypothesis is true.
H_0

<u>Type II error</u> : the error of accepting H_0 when H_0 is false.

Ex] find type I and II error probabilities if the test "decision rule" is as follows:

Reject H_0 and accept H_1 if 10 flips yield fewer than 7 heads. (regular coin)

Accept H_0 and reject H_1 if 10 flips yield at least 7 heads. (special coin) Let <u>S_n be # of heads in</u> n tosses. we have $n=10$.

$\alpha =$ P(type I error)

$B =$ P(type II error)

$\alpha =$ P(test will reject H_0 when H_0 is true)

$= P(S_n < 7 \text{ when } \theta = \theta_0 = \frac{3}{4})$ Probability of less than 7 successes in 10 trials with success prob. = 3/4.

$$= \sum_{k=0}^{6} \binom{10}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{10-k}$$

$= [P(S_n=0) + P(S_n=1) + \dots + P(S_n=6)] \approx .2241$

$B =$ P(type II error)

$=$ P(test will accept H_0 when H_0 is false)

$= P(S_n \geq 7 \text{ when } \theta = \theta_1 = \frac{1}{2})$

we are looking for probability of 7 or more successes in 10 trials with success probability equal to $\frac{1}{2}$.

$$= P(S_n = 7) + P(S_n = 8) + P(S_n = 9) + P(S_n = 10)$$

$$= \sum_{k=7}^{10} \binom{10}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{10-k} \approx 0.1719$$

Suppose that the test of hypothesis is carried out 10,000 times.

Then it is very likely the collector will make type I errors approx. $\alpha \times 10{,}000 = .2241 \times 10{,}000 = 2{,}241$ times

and that he will make type II errors approx $\beta \times 10{,}000 = .1719 \times 10{,}000 = 1719$ times

$\Rightarrow$ He will only make the correct decision approx. $10{,}000 - 2{,}241 - 1{,}719 = 6{,}040$ times.

Ex) Suppose the test is altered to involve more flips. In particular, let $n = 100$ be the # of flips.

"Decision rule":

Reject H_0 and accept H_1 if 100 flips yield fewer than 65 heads.

Accept H_0 and reject H_1 if 100 flips yield at least 65 heads.

S_n = # of heads in $n=100$ flips.

Let $Z = \dfrac{S_n - n\theta}{\sqrt{n\theta(1-\theta)}}$ where θ is the success probability

(i.e. $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$) normal approx to the binomial

Since n large, $Z \sim N(0,1)$ approx. std normal

$$= P(\text{type I error})$$

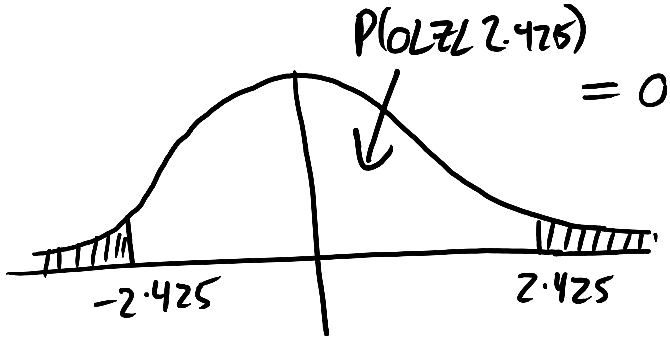
$$\alpha = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

$$= P(\underset{\text{binom}}{S_n} < 65 \text{ when } \theta = \theta_0 = \tfrac{3}{4})$$

$$\approx P(\underset{\text{normal}}{S_n} < 64.5 \text{ when } \theta = \theta_0 = \tfrac{3}{4})$$

$$= P\left(\frac{S_n - n\theta}{\sqrt{n\theta(1-\theta)}} < \frac{64.5 - \overset{= np = \frac{3}{4}\cdot 100}{75}}{\sqrt{18.75}}\right)$$

$$= P(Z < -2.425) = 0.5 - A(2.425)$$



$= 0.5 - P(0 < Z < 2.425)$

$= 0.008$

$\underline{\beta > \alpha}$

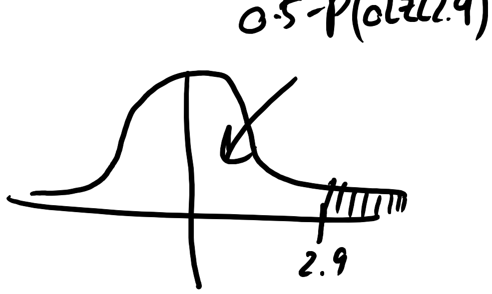
$\beta = P(\text{type II error})$

$= P(\text{test will accept } H_0 \text{ when } H_0 \text{ is false})$

$= \underset{\text{binomial}}{P(S_n \geq 65} \text{ when } \theta = \theta_1 = \tfrac{1}{2})$

$\approx \underset{\text{normal}}{P(S_n \geq 64.5} \text{ when } \theta = \theta_1 = \tfrac{1}{2})$

$= P\left(\dfrac{S_n - n\theta}{\underbrace{\sqrt{n\theta(1-\theta)}}_{\sqrt{100 \cdot \frac{1}{2} \cdot \frac{1}{2}}}} > \dfrac{64.5 - 50}{5}\right)$



$= P(Z > 2.9) = 0.02$

$= 0.5 - .498\ldots$

Notice that the second decision rule has much smaller error probabilities.

Test for means of normal populations (or when $n \geq 30$) with known variances.

Test statistic:

$$z = \frac{\bar{X} - m_0}{\sigma / \sqrt{n}}$$

significance level

$$\alpha = P(\text{type I error})$$

$H_0: \; m = m_0$ (null hypothesis)

	Decision rule
$H_1: \; m \neq m_0$	$\lvert z \rvert > z_{\alpha/2}$ (two sided test)
$H_1: \; m > m_0$	$z > z_\alpha$ } one sided
$H_1: \; m < m_0$	$z < -z_\alpha$ } one sided

Ex) Suppose for some situation,

$H_0: \; m = 3$ (null Hypothesis)

$H_1: \; m < 3$ (alternate Hypothesis)

given, significance level

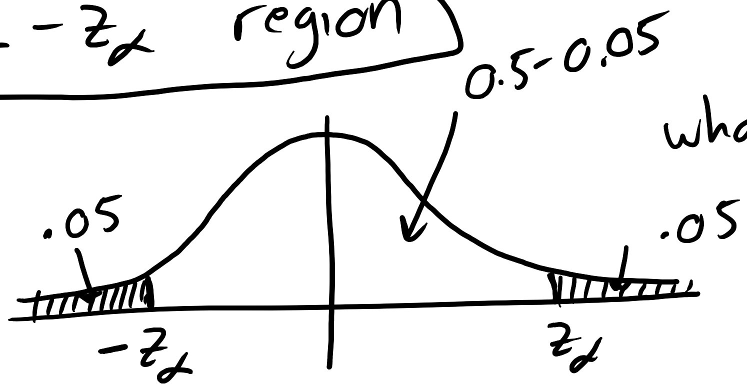
Let $\bar{X} = 2.8$, $n = 10$, $\boxed{\sigma = .25}$, $\boxed{\alpha = .05}$

(that is from 10 samples, the sample mean is 2.8).

$\Rightarrow z_\alpha = 1.645$

rejection region

$\boxed{z < -z_\alpha \text{ region}}$



what is z_α

z_α corresponds to $P(0 < z < z_\alpha) = 0.5 - .05$
$= .45$

$\Rightarrow$ use table to find $\underline{z_\alpha = 1.65}$

Thus, we would reject H_0 if:

$$z < -z_\alpha = -1.65$$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{2.8 - 3}{.25/\sqrt{10}} = -2.53$$

At the .05 significance level, we would reject H_0.

<u>Tests for means of normal populations with unknown variances</u>

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$ (t-distribution statistic with sample std deviation is used)

$H_0: \mu = \mu_0$

Decision rule

$H_1: \mu \neq \mu_0$ $\quad |t| > t_{\alpha/2, n-1}$

$H_1: \mu > \mu_0$ $\quad t > t_{\alpha, n-1}$

$H_2: \mu < \mu_0$ $\quad t < -t_{\alpha, n-1}$

Ex) Owner of restaurant tells perspective buyer that # of customers per day is about 100.

Sample $n = 9$ is chosen. $\Rightarrow \bar{x} = 95, \quad s = 10$ } buyer wants to check over 9 days

Should the buyer reject owners claim at .05 level?

$H_0: \mu = 100; \quad H_1: \mu < 100$

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{95 - 100}{10/\sqrt{9}} = -1.5$$

Reject H_0 if $t < -t_{\alpha, n-1}$

$\alpha = .05 \Rightarrow \underbrace{t_{\alpha, n-1} = t_{.05, 8} = 1.860}_{\text{from table}}$

$T = -1.5$ is <u>not</u> less than -1.860

we cannot reject H_0!