A) In a random sample, 136 of 400 people had side effects to vaccine. Construct a 95% confidence interval for true proportion having side effect to the vaccine.

Here X is r.v. counting # of people with side effects in n samples. We like to construct a 95% confidence interval for p (success probability). p is also called the true proportion (of successes).

In our sample, $n=400$, $X=136$

sample proportion $= \hat{p} = \frac{136}{400} = 0.34$

Since $n=400$ (large sample),

$$Z = \frac{X-np}{\sqrt{np(1-p)}} \sim N(0,1)$$

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1-\alpha$$

$$\Rightarrow P\left(-z_{\alpha/2} < \frac{X-np}{\sqrt{np(1-p)}} < z_{\alpha/2}\right) = 1-\alpha$$

Consider one side at a time:

$$\frac{X - np}{\sqrt{np(1-p)}} < z_{\alpha/2} \Rightarrow X - np < z_{\alpha/2}\sqrt{np(1-p)}$$

$$\Rightarrow X < np + z_{\alpha/2}\sqrt{np(1-p)}$$ divide by $n>0$:

$$\Rightarrow \frac{X}{n} < p + z_{\alpha/2}\sqrt{\frac{np(1-p)}{n^2}}$$ replace $\frac{X}{n}$ by $\hat{p}$ and $p(1-p)$ by $\hat{p}(1-\hat{p})$

$$\Rightarrow p > \hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

other inequality yields $p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

is an approx. $(1-\alpha)\cdot 100\%$ CI for true proportion p.

We have $n=400$, $\hat{p} = \frac{136}{400} = 0.34$, $1-\hat{p} = 0.36$

Also, 95% confidence interval yields

$\alpha = 0.05$ $(1-\alpha = 0.95) \Rightarrow \frac{\alpha}{2} = 0.025$

$P(|Z| < z_{\alpha/2}) = 1-\alpha = 0.95$

$P(0 < Z < Z_{\alpha/2}) = \frac{1}{2}(0.95) = 0.475$

Via std normal area table, can find that

$Z_{0.025} = 1.96$

(since area between $z=0$ and $z=1.96$ under std. normal curve is about 0.475).

Plugging in, we get the 95% CI:

$$0.34 - 1.96\sqrt{\frac{0.34(0.66)}{400}} < p < 0.34 + 1.96\sqrt{\frac{0.34(0.66)}{400}}$$

$\Rightarrow 0.29 < p < 0.39$ (with 95% confidence)

↑ true population proportion of people feeling side effects to vaccine.

B) For 12 areas of same size ($n=12$),

$\bar{X} = 66.3$ (sample mean) and

$s = 8.4$ (sample std deviation)

construct 95% CI for true mean μ.

Suppose population is normally distributed.

notice: population standard deviation is not known. Even though we know the population is normally distributed, since n is small we use here the student t-distribution.

In particular, $T = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ is a random variable having a t-distribution with $n-1$ degrees of freedom.

$$P(-t_{\alpha/2,n-1} < T < t_{\alpha/2,n-1}) = 1 - \alpha$$

substituting the above for T, yields the confidence interval:

$$\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$$

with $(1-\alpha)\cdot 100\%$ confidence for population mean μ.

In our case, $\alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$

$n = 12 \Rightarrow n-1 = 11$

$t_{\frac{\alpha}{2},n-1} = t_{0.025,11} = 2.201$ (from t-distribution table)

Since $\bar{x} = 66.3$ and $s = 8.4$, we get:

$$66.3 - 2.201 \frac{8.4}{\sqrt{12}} < \mu < 66.3 + 2.201 \frac{8.4}{\sqrt{12}}$$

$$\Rightarrow 61.0 < \mu < 71.6$$

We can assert with 95% confidence that the interval from 61 to 71.6 min contains the true average drying time of the paint.