A) $\sigma = 20$, mean not known, $n = 100$

We use sample mean $\bar{x}$ to estimate population mean μ.
We need to find $P(|\bar{x}-\mu| < 3)$, which is the probability that the error in estimation of μ by $\bar{x}$ will be less than 3.

Recall $P(-z_{\alpha/2} < z < z_{\alpha/2}) = 1-\alpha = P(|z| < z_{\alpha/2}) = 1-\alpha$

Since $n \geq 30$, CLT applies so that $\bar{x}$ is normally distributed.
$\bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}}) \Rightarrow \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ std normal

$\Rightarrow z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \Rightarrow P\left(\frac{|\bar{x}-\mu|}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1-\alpha$

$\Rightarrow P\left(|\bar{x}-\mu| < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$

$\Rightarrow z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 3 \Rightarrow z_{\alpha/2}\frac{20}{\sqrt{100}} = 3 \Rightarrow z_{\alpha/2} = \frac{3}{2} = 1.50$

$P(|z| < z_{\alpha/2}) = 1-\alpha = P(|z| < \frac{3}{2}) = 2P(0 < z < \frac{3}{2})$

We can find $P(0 < z < \frac{3}{2})$ from table to be 0.4332

$\Rightarrow P(|x-\mu| < 3) = 1-\alpha = 2(0.4332) = 0.8664 \simeq 0.87$

(B) $P(X=0) = 0.20$; $P(X=1) = 0.20$; $P(X=2) = 0.40$; $P(X=3) = 0.20$
Probabilities add to 1 so this is a complete probability distribution
$\mu_X = E[X] = \sum x p(x) = 0(0.20) + 1(0.20) + 2(0.40) + 3(0.20)$
$= 0 + 0.20 + 0.80 + 0.60 = 1.60$
$\sigma_X^2 = E[(x-\mu)^2] = \sum (x-\mu)^2 p(x)$

$$\sigma_X^2 = (0-1.60)^2(0.20) + (1-1.60)^2(0.20) + (2-1.60)^2(0.40)$$
$$+ (3-1.60)^2(0.20) = 1.04$$

$\Rightarrow \sigma_X = 1.0198$

(c) $P(X=0)=0.80$, $P(X=1)=0.12$, $P(X=2)=0.05$
$P(X=3) = 0.03$

$\Rightarrow$ P(more than 100 dead bugs in 50 bags)
$= P(\bar{X} > 2)$ (more than 2 bugs per bag on avg)

Since $n=50$ ($n \geq 30$), the CLT applies.

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$P(\bar{X}>2) = P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > \frac{2-\mu}{\sigma/\sqrt{n}}\right) = P\left(Z > \frac{2-\mu}{\sigma/\sqrt{n}}\right)$$

To evaluate this, we must find μ and σ, as before.

$$\Rightarrow E[X] = \mu = \sum x p(x) = 0(0.80) + 1(0.12) + 2(0.05) + 3(0.03)$$
$$= 0.12 + 0.10 + 0.09 = 0.31 \quad \text{(expected \# of bugs per bag)}$$

Notice: $E[X] = 0.31$ (which can never be the # of bugs in a bag; expected value does not need to be an outcome).

$$\Rightarrow \sigma^2 = E[(x-\mu)^2] = \sum (x-\mu)^2 p(x) =$$
$$= (0-0.31)^2(0.80) + (1-0.31)^2(0.12) + (2-0.31)^2(0.05)$$
$$+ (3-0.31)^2(0.03) = 0.4939$$

$\Rightarrow \sigma \approx 0.703$

$$P\left(Z > \frac{2-\mu}{\sigma/\sqrt{n}}\right) = P\left(Z > \frac{2-0.31}{0.703/\sqrt{150}}\right) = P(Z > 16.99)$$

$$= 1 - P(Z \le 16.99) = 1 - [P(-\infty < Z < 0) + P(0 < Z < 16.99)]$$

$$= 1 - [0.5 + 0.499] \le 0.01$$

Thankfully, the probability of so many bugs in the spinach is very small. Wash and eat.

D) Toss 300 fair coins. Find the probability that we get more than 160 heads and that we get less than 120 heads.

X binomial random variable, counts # of heads in n tosses

$p = 0.5$, $q = 1 - p = 0.5$

$$P(X > 160) = 1 - P(X \le 160) = 1 - [P(X=0) + P(X=1) + \ldots + P(X=160)]$$

= 1 - pbinom(160, size=300, prob=0.5)

$= 1 - 0.887 = 0.113$

$P(X < 120) = P(X \le 119) =$ pbinom(119, size=300, prob=0.5)

$= 0.0002$ (quite small!)

since $n=300$ (lots of trials), we can approximate with a normal distribution

$$Z = \frac{X - np}{\sqrt{npq}} \sim N(0,1) \quad \text{(approx std. normal)}$$

$$Z = \frac{X - (300)(0.5)}{\sqrt{300 \cdot 0.5^2}} = \frac{X - 150}{8.66}$$

$P(X_{binom} > 160) \approx P(X_{norm} > 159.5) = P\left(Z > \frac{159.5 - 150}{8.66}\right)$

$= P(Z > 1.097) =$ pnorm(159.5, mean=150, sd=8.66, lower.tail=FALSE) = 0.137

This is an acceptable approximation.

$P(X_{binom} < 120) \approx P(X_{normal} < 120.5)$

= pnorm(120.5, mean=150, sd=8.66) = 0.003

which is almost the same as the binomial distribution calculation.

For the poisson distribution we set $\lambda = np = 150$.
We don't expect a good approx because we are not dealing with rare events (p is small).

$P(X_{binom} > 160) \approx P(X_{poisson} > 160) = P(X_{poisson} \geq 161)$

= ppois(160, lambda=150, lower=FALSE) = 0.195

$P(X_{binom} < 120) \approx P(X_{poisson} < 120)$

= ppois(120, lambda=150, lower=TRUE) = 0.0065

The approximations are not very accurate but do give acceptable results, despite the fact that p is not small.