Solutions to the Final Exam

(1) (a) $P(A) = 0.86$, $P(B) = 0.35$

$P(A \cap B) = 0.29$

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.86 + 0.35 - 0.29 = 0.92$

(b) $P(A' \cap B') = P\left[[A \cup B]'\right]$ de Morgans Law

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{7}\right] = 0.31$

(2) (a) $H_0: p = 0.01$; $\alpha = 0.05$

$H_1: p > 0.01$

$n = 80$, 2 defective out of 80

$\hat{p} = \frac{2}{80} = 0.025$

Decision rule:

$z = \frac{n\hat{p} - np_0}{\sqrt{np_0q_0}}$

reject H_0 if

$z > z_\alpha = z_{0.05} = 1.645$

$$\Rightarrow z = \frac{2 - 80(0.01)}{\sqrt{80(0.01)(0.99)}} \approx 1.348$$

Since $z < z_\alpha$, we cannot reject H_0.
Notice that this test is likely not very good since the number of samples n is not large enough.

"only approx here"

$$z = \frac{n\hat{p} - np_0}{\sqrt{np_0 q_0}} \sim N(0,1)$$

(b) See HW #12 solution

$\alpha =$ P(type I error) = P(reject H_0 when H_0 true)

$= P(\bar{x} > 1.38 \text{ when } \mu = 1)$

$= P\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} > \frac{1.38 - 1}{.5/\sqrt{9}}\right) = P(z > 2.33) = 0.01$

$\beta =$ P(type II error) = P(accept H_0 when H_0 false)

= P(accept H_0 when H_1 true)

$= P(\bar{x} \leq 1.38 \text{ when } \mu = 2) = P\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq \frac{1.38 - 2}{.5/3}\right)$

$= P(z \leq -3.67) \approx 0.001$

(3) (a) Two rolls of dice each labeled 1-6

Total of $6 \times 6 = 36$ possible outcomes.

How to get sum=7 on two rolls:

$$\left\{ \begin{matrix} 1 \text{ and } 6 & 3 \text{ and } 4 & 2 \text{ and } 5 \\ 6 \text{ and } 1 & 4 \text{ and } 3 & 5 \text{ and } 2 \end{matrix} \right\}$$

$$P(\text{sum}=7) = \frac{6}{36} = \frac{1}{6}$$

(b) X counts # of times sum=7 in n trials. In each trial, two dice are rolled. X follows binomial distribution. $n=3$

$$P(X \text{ at least } 1) = P(X \geq 1) = 1 - P(X=0)$$

$$P(X \geq 1) = 1 - \left[\binom{3}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 \right] \simeq 0.42$$

(c) $p = 0.005$; $n = 1000$

$$P(X>2) = 1 - P(X \leq 2) = 1 - \underbrace{[P(X=0) + P(X=1) + P(X=2)]}_{\text{mutually exclusive events}}$$

$$\lambda = np = 1000(0.005) = 5$$

$$P(X_{binom} > 2) \approx 1 - \sum_{k=0}^{2} P(X_{poisson} = k)$$

$$P(X_p = 0) = \frac{5^0 e^{-5}}{0!} \qquad P(X_p = 2) = \frac{5^2 e^{-5}}{2!}$$

$$P(X_p = 1) = \frac{5^1 e^{-5}}{1!}$$

$\vdots$

(4) $P(X=0) = 0.65$, $P(X=1) = 0.18$
$P(X=2) = 0.10$, $P(X=3) = 0.07$

$$E[X] = \sum x p(x) = 0(0.65) + 1(0.18) + 2(0.10) + 3(0.07) \approx 0.59$$

$$\sigma^2 = \sum (x-\mu)^2 p(x)$$
$$= [0-0.59]^2(0.65) + [1-0.59]^2(0.18) + [2-0.59]^2(0.10) + [3-0.59]^2(0.07)$$

$\sigma^2 \approx .8619 \quad \Rightarrow \sigma \approx 0.928 \quad ; \quad n=50$

$$P(\bar{X} > 1) = P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > \frac{1-\mu}{\sigma/\sqrt{n}} \right)$$

$$\overset{CLT}{\approx} P\left(Z > \frac{1 - 0.59}{0.93/\sqrt{50}} \right) = P(Z > 3.11) \approx 0.001$$

The probability of more than 50 remains of bugs in 50 bags is very small.

(5) 10 tests $\Rightarrow n=10$

$\bar{X} = 65$ and $s=8$

small sample estimation, σ not known.
No reason to assume population is not approximately normally distributed.
$\Rightarrow$ Use student-t distribution

90% CI:

$\alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05 \Rightarrow t_{\alpha/2, n-1} = t_{0.05, 9} = 1.833$

$\bar{X} - 1.833 \frac{s}{\sqrt{n}} < \mu < \bar{X} + 1.833 \frac{s}{\sqrt{n}}$ is the 90% CI for population mean μ.

$\Rightarrow 60.36 < \mu < 69.64$ (90% CI)

95% CI:

$\alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$

$t_{\frac{\alpha}{2}, n-1} = t_{0.025, 9} = 2.262$

$\Rightarrow 59.28 < \mu < 70.72$ (95% CI)

(6) We use $\bar{x}$ as an estimator for μ.

$P(|\bar{x} - \mu| < 3) = ?$

$\Rightarrow P(|\bar{x} - \mu| < 3) = P(-3 < \bar{x} - \mu < 3)$

$= P\left(\frac{-3}{\sigma/\sqrt{n}} < z < \frac{3}{\sigma/\sqrt{n}}\right)$

We use $\sigma \approx s = 5$ and $n = 50$

$= P\left(\frac{-3}{5/\sqrt{50}} < z < \frac{3}{5/\sqrt{50}}\right)$

$\approx P(-4.24 < z < 4.24) = 2P(0 < z < 4.24)$

$\approx 2 \times 0.4999 \approx 0.9999$

Almost perfect chance that $\bar{x}$ will be within 3 units of μ.

(7)

$$\text{sort}(x) = \{\overset{1}{8}, \overset{2}{17}, \overset{3}{18}, \overset{4}{18}, \overset{5}{18}, \overset{6}{19}, \overset{7}{20}, \overset{8}{21}, \overset{9}{22}, \overset{10}{22}, \overset{11}{25}, \overset{12}{32}\}$$

min $= 8$, max $= 32$

$Q_1 = 25^{th}$ %tile $\Rightarrow L_{25} = \left(\frac{25}{100}\right)12 = 3$

$$Q_1 = \frac{\text{sort}(x)_3 + \text{sort}(x)_4}{2} = 18$$

$Q_2 = 50^{th}$ %ile $\Rightarrow L_{50} = \frac{1}{2} \times 12 = 6$

$$Q_2 = \text{median} = \frac{\text{sort}(x)_6 + \text{sort}(x)_7}{2} = 19.5$$

$Q_3 = 75^{th}$ %ile $\Rightarrow L_{75} = \left(\frac{75}{100}\right) \times 12 = 9$

$$Q_3 = \frac{\text{sort}(x)_9 + \text{sort}(x)_{10}}{2} = 22$$

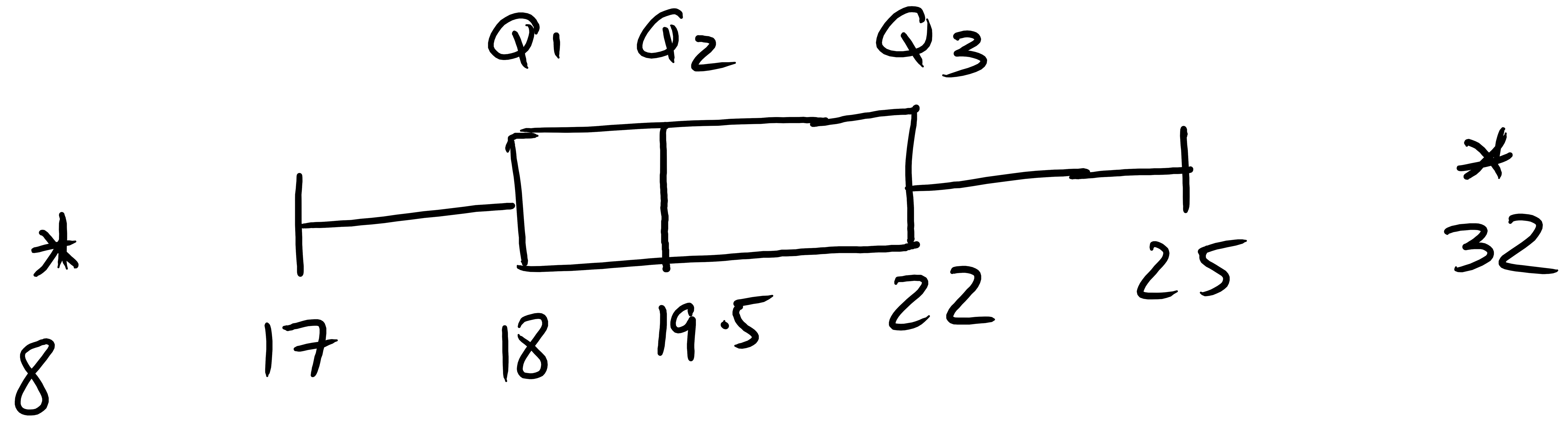
$IQR = Q_3 - Q_1 = 22 - 18 = 4$

outliers: $< Q_1 - 1.5\,IQR = 18 - 6 = 12$

$> Q_3 + 1.5\,IQR = 22 + 6 = 28$

8 and 32 are the outliers.

modified boxplot of data:



(8) (a) $n = 500$ tosses, $X = \#$ of heads

X follows binomial distribution

$E[X] = np = 500(0.5) = 250$

$\sigma^2(x) = npq = 500(.5)(.5) = 125$

(b) $P(250 - 12 < X_{binom} < 250 + 12)$

$= P(238 < X_{binom} < 262)$

$\approx P(237.5 < X_{normal} < 262.5)$

$= P\left(\frac{237.5 - \mu}{\sigma} < Z < \frac{262.5 - \mu}{\sigma}\right)$

$= P\left(\frac{237.5 - 250}{11.18} < Z < \frac{262.5 - 250}{11.18}\right)$

$= P(-1.12 < z < 1.12)$

$= 2P(0 < z < 1.12) \approx 2(0.344) = 0.688$

(9) $H_0: \mu_A - \mu_B = 0$ $\alpha = 0.01$

$H_1: \mu_B - \mu_A > 0$

$n_A = 50, \quad n_B = 60$ (samples from 2 populations)

$\bar{X}_A = 10, \quad \bar{X}_B = 12$ in grams per glass

$S_A = 1, \quad S_B = 1.5$

$$z = \frac{(X_B - X_A) - (\mu_B - \mu_A)_{\text{per } H_0}}{\sqrt{\frac{S_A^2}{n_A} + \frac{S_B^2}{n_B}}}$$

$$= \frac{(12-10) - 0}{\sqrt{\frac{1.5^2}{60} + \frac{1^2}{50}}} \approx \frac{2}{0.24} = 8.34$$

Decision rule: reject H_0 if $z > z_\alpha = z_{0.01} = 2.33$

At $\alpha = 0.01$ confidence level, we reject H_0 in favor of H_1.

$P(\text{type I error}) = \alpha = 0.01$

(10) (a) $P(\text{3 aces amongst 5 cards})$

$= P(\text{3 aces, 2 other cards}) = \dfrac{(4C3)(49C2)}{(52C5)}$

(b) $P(\text{3 kings, 2 queens}) = \dfrac{(4C3)(4C2)}{(52C5)}$

(c) $P(A|S) = \dfrac{P(A \cap S)}{P(S)} = \dfrac{0.05}{0.1} = 0.5 \neq P(A)$

The event are dependent.